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# Moore bound for mixed networks[☆]

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**Abstract**

Mixed graphs contain both undirected as well as directed links between vertices and therefore are an interesting model for interconnection communication networks. In this paper, we establish the Moore bound for mixed graphs, which generalizes both the directed and the undirected Moore bound.


*Keywords:* Mixed Moore graph; Mixed Moore bound; $T$-graph; Tied graphs

## 1. Introduction

Graphs considered in this paper are finite and *mixed*, i.e., they may contain (directed) *arcs* as well as (undirected) *edges*.

Let $v$ be a vertex of a mixed graph $G$. Denote by $d^-(v)$ (respectively, $d^+(v)$) the number of arcs incident to (respectively, from) $v$. We also call $d^-(v)$ (respectively, $d^+(v)$) the number of in-neighbours (respectively, out-neighbours) of $v$. Denote by $id(v)$ (respectively, $od(v)$) the sum of the number of arcs incident to (respectively, from) $v$ and the number of edges incident with $v$. Denote by $r(v)$ the number of edges incident with $v$ (i.e., the *undirected degree* of $v$). $G$ is said to be *locally regular* if, for each vertex $v$ of $G$, $od(v) = id(v)$ (i.e., $d^-(v) = d^+(v)$). $G$ is said to be *regular* of degree $d$ if $od(v) = id(v) = d$ for every vertex $v$ of $G$. A regular graph $G$ of degree $d$ is said to be *totally regular* with undirected degree $r$ and *directed degree* $z = d - r$ if, for every vertex $v$ of $G$, we have $r(v) = r$.

Let $G$ be a mixed graph and let us denote by $E(G)$ (respectively, $A(G)$) the set of all edges (respectively, all arcs) of $G$. Conventionally, if $A(G) = \emptyset$ then $G$ is simply an undirected graph and if $E(G) = \emptyset$ then $G$ is a directed graph.

$G$ is said to be a *proper* mixed graph if $G$ contains at least one arc and at least one edge.

Let $uv$ be an edge in $G$. For $1 \le l \le k$, where $k$ is the diameter of $G$, by $N_l^*(u)$ and $T_l^*(u)$, we denote the multiset of all vertices reachable from $u$ using mixed trails of length exactly $l$ and at most $l$, respectively. We use $N^*(u)$ as a shorter notation for $N_1^*(u)$. For a subset $S$ of the vertex set of a mixed graph, by $N^*(S)$ we mean the multiset of vertices reachable from the vertices of $S$ using mixed trails of length exactly 1.

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Note that if $E(G) = \emptyset$ then $N^*(u)$ is usually denoted by $N^+(u)$ (the set of *out-neighbours* of $u$) and if $A(G) = \emptyset$ then $N^*(u)$ is usually denoted by $N(u)$. In the case of a proper mixed graph where $E(G) \neq \emptyset$ and $A(G) \neq \emptyset$, $N^*(u) = N^+(u) \bigcup N(u)$.

For other relevant graph-theoretic concepts, please refer to [11].

**Definition 1.** A mixed graph $G$ is said to be a *T-graph* if, for each ordered pair $(u, v)$ of vertices of $G$ (possibly $u = v$), there exists in $G$ exactly one mixed trail from $u$ to $v$ of length less than or equal to the diameter.

Note that $T$-graphs were also studied under the name *strongly geodetic graphs* [2,4,10]. Note also that for every vertex $u$ in a $T$-graph of diameter $k$, both $N_l^*(u)$ and $T_l^*(u)$ $(1 \leq l \leq k)$ are sets (i.e., every element has multiplicity one).

The following theorem was proved by Bosák (Theorem 1 in [3]).

**Theorem 1** (*[3]*). *Every $T$-graph is either an undirected tree or a regular graph with a finite diameter.*

A regular $T$-graph of a finite diameter is called a *tied graph* [2,4].

By the definition of mixed graphs, directed and undirected tied graphs are special cases of mixed tied graphs, where the graphs under consideration admit either only arcs or only edges.

The order $n_{d,k}$ of an undirected graph of maximum degree $d$ and diameter $k$ is bounded above by the *Moore bound*

$$n_{d,k} \leq M_{d,k} = 1 + d + d(d-1) + \cdots + d(d-1)^{k-1}. \tag{1}$$

If equality holds in Eq. (1) then the graph is called a *Moore graph*. For $k \geq 3$ and $d \geq 3$, Moore graphs do not exist [1,6,7].

In a directed graph, $od(v)$ is often called the *out-degree* of the vertex $v$. Similarly, the order $n^*_{d,k}$ of a directed graph of maximum out-degree $d$ and diameter $k$ is bounded above by the *directed Moore bound*

$$n^*_{d,k} \leq M^*_{d,k} = 1 + d + d^2 + \cdots + d^k. \tag{2}$$

If equality holds in Eq. (2) then the digraph is called a *Moore digraph*. Similar to the undirected case, Moore digraphs do not exist for $d > 1$ and $k > 1$ [5,10].

For more information concerning extremal problems related to Moore bounds, both directed and undirected, refer to the survey by Miller and Širáň [8]. It was shown in [10] that directed tied graphs of maximum out-degree $d$ are regular of degree $d$. Thus, with a simple counting argument, one can verify the following.

**Theorem 2.** *Let $G$ be a digraph. Then the following two assertions are equivalent.*

(i) *$G$ is a Moore digraph of maximum out-degree $d$, diameter $k$ and order $M^*_{d,k}$;*
(ii) *$G$ is a directed tied graph of maximum out-degree $d$, diameter $k$ and order $M^*_{d,k}$.*

Similarly, in the undirected case, Bosák, Kotzig and Znám [4] have proved

**Theorem 3** (*[4]*). *Let $G$ be an undirected graph. Then the following two assertions are equivalent.*

(i) *$G$ is an undirected Moore graph of maximum degree $d$, diameter $k$ and order $M_{d,k}$;*
(ii) *$G$ is an undirected tied graph of maximum degree $d$, diameter $k$ and order $M_{d,k}$.*

Theorems 2 and 3 show that Moore digraphs (respectively, graphs) are directed (respectively, undirected) tied graphs.

However, in the case of proper mixed graphs, such definitive equivalence between the two classes, namely "proper mixed Moore graphs" and "proper mixed tied graphs", is by no means obvious.

Theorem 1 shows the regularity of all $T$-graphs. However, it is not strong enough to establish a numerical bound on the number of vertices of $T$-graphs. In the next section, we prove a stronger result of Theorem 1, showing that all $T$-graphs are also totally regular. This result helps us to establish the "mixed Moore bound" for mixed graphs, which generalizes both directed and undirected Moore bounds.

## 2. The mixed Moore bound

Note that we only need to consider 'proper mixed' tied graphs. We observe the following.

**Observation 1.** *Let $G$ be a strongly connected proper mixed graph of degree $d \geq 1$ and diameter $k \geq 2$. Then there exists a vertex $u \in V(G)$ such that $r(u) \geq 1$ and $z(u) \geq 1$.*

**Lemma 1.** *Let $G$ be a proper mixed tied graph of diameter $k \geq 2$ and let $uv$ be an edge of $G$ such that the directed degree of $u$ or $v$ is nonzero. Then $r(v) = r(u)$.*

**Proof.** It follows from Theorem 1 that $G$ is regular. Let $z, z'$ be the directed degrees of $u$ and $v$, respectively. By our assumption, we have $z > 0$ or $z' > 0$. We shall show that $z' = z$. Let us assume, without loss of generality, that $0 \leq z' < z$. We denote by $u_1, \ldots, u_z$ the out-neighbours of $u$. Let us consider vertex $u_1$. It is easy to see that $v$ is at a distance $k$ from $u_1$, since otherwise there would be at least two different mixed trails of length at most $k$ from $u$ to $v$. Let $T$ be the trail of length $k$ from $u_1$ to $v$ and $v'_1$ be the vertex preceding $v$ on $T$. Then $v'_1 v$ is an arc from $v'_1$ to $v$ in $G$ because if $v'_1 v$ were an edge then there would be at least two different mixed trails of lengths at most $k$ from $u$ to $v'_1$. By repeating the argument for the other $z - 1$ out-neighbours of $u$, we obtain additional $z - 1$ in-neighbours $v'_2, \ldots, v'_z$ of $v$. Clearly, the vertices in the set $S = \{v'_1, \ldots, v'_z\}$ must be all different since otherwise vertex $u$ could reach at least one vertex in $S$ using two different mixed trails of length $k$. As a consequence, we have $z' \geq z$. But this contradicts our assumption. Therefore, $z' = z$ and, by Theorem 1, the result follows. □

**Lemma 2.** *Let $G$ be a proper mixed tied graph of diameter $k \geq 2$ and let $uu_1$ be an arc from $u$ to $u_1$ in $G$. Then $r(u_1) = r(u)$.*

**Proof.** Since $G$ is regular (see Theorem 1), let $z$ (respectively, $z'$) be the directed degree of $u$ (respectively, $u_1$). Let us first consider the case when $z = 1$. This means that $z' \geq 1$. Let $N(u) = \{u_2, \ldots, u_d\}$. Since there is already a trail of length two from each $u_i$ $(2 \leq i \leq d)$ to $u_1$, there is no arc incident from vertices in $\bigcup_{i=2}^{d} T^*_{k-1}(u_i)$ to $u_1$. In addition, there is no arc incident from $T^*_{k-1}(u_1)$ to $u_1$. Therefore, $z' = 1$ and, consequently, $r(u) = r(u_1)$.

Let us now suppose that z is at least two and let $N^+(u) = \{u_1, u_2, \ldots, u_z\}$ be the out-neighbours of $u$. It is not difficult to see that the distance from $u_i$ $(2 \leq i \leq z)$ to $u_1$ must be exactly $k$. This is because otherwise there would be at least two different mixed trails of lengths not exceeding $k$ from $u$ to $u_1$. Moreover, let $u_{i1}$ $(2 \leq i \leq z)$ be the vertex preceding $u_1$ on the trail of length $k$ from $u_i$ to $u_1$. We can see that $u_{i1}u_1$ must be an arc since otherwise there would be at least two different mixed trails of lengths not exceeding $k$ from $u$ to $u_{i1}$, namely $u \to u_i \cdots u_{i1}$ and $u \to u_1 - u_{i1}$, which is impossible. Obviously, the $z$ vertices in set $S = \{u, u_{21}, \ldots, u_{z1}\}$ are distinct. Otherwise, there would be at least two different mixed trails of lengths at most $k$ from $u$ to a vertex in $S$. Thus, $z' \geq z$.

We shall now show that $z' = z$. Let us suppose, by contradiction, that $z' \geq z + 1$. Then there exists some in-neighbour $v_1$ of $u_1$ such that $v_1 \in (V(G) \setminus S)$. Let $N(u) = \{u_{z+1}, \ldots, u_d\}$. Since the diameter of $G$ is $k$, $v_1$ must occur in $\bigcup_{j=1}^{d} T^*_{k-1}(u_j)$. Clearly, $v_1 \notin T^*_{k-1}(u_1)$. But then if $v_1 \in T^*_{k-1}(u_j)$ for some $2 \leq j \leq z$, there would exist at least two different mixed trails of lengths not exceeding $k$ from $u_j$ to $u_1$, namely $u_i \ldots u_{j1} \to u_1$ and $u_i \ldots v_1 \to u_1$. This clearly contradicts Definition 1. So $v_1$ must occur in a set $T^*_{k-1}(u_h)$ $(z + 1 \leq h \leq d)$. This, however, is also impossible because then there would be at least two different mixed trails of maximum length $k$, namely $u_h - u \to u_1$ and $u_h \ldots v_1 \to u_1$, from $u_h$ to $u_1$.

Therefore, by Theorem 1, the lemma follows. □

Combining the results of Theorem 1, Lemmas 1 and 2 we derive the following

**Theorem 4.** *Let $G$ be a tied graph of order $n$, degree $d$ and diameter $k \geq 2$. Then $G$ is a totally regular graph. If $z$ is the directed degree of $G$ and $r$ is the undirected degree of $G$, then $n = M_{z,r,k} = 1 + (z + r) + z(z + r) + r(z + r - 1) + \cdots + z(z + r)^{k-1} + r(z + r - 1)^{k-1}$.*

**Corollary 1.** *Let $G$ be a mixed graph of diameter $k$, maximum degree $d$ and maximum out-degree $z$. Let $r = d - z$. Then the order $n$ of $G$ is bounded by*

$$n \leq M_{d,z,k} = 1 + d + zd + r(d - 1) + \cdots + zd^{k-1} + r(d - 1)^{k-1}. \tag{3}$$

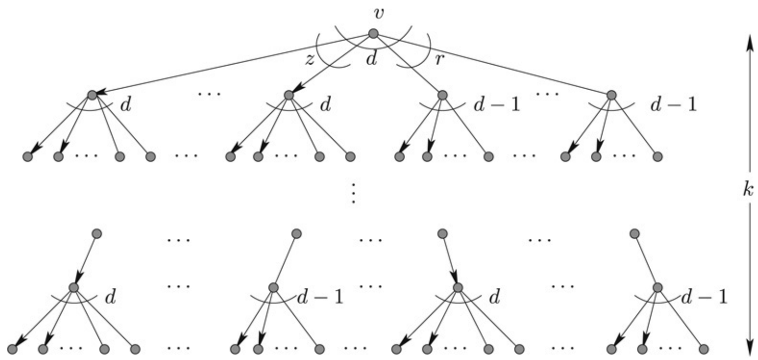


Fig. 1. Moore bound for mixed graphs.

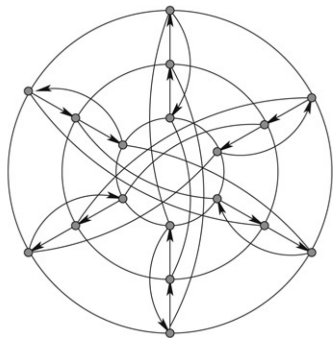

Fig. 2. The Bosák graph.

Fig. 1 illustrates the Moore bound for mixed graphs of diameter $k$, maximum degree $d$ and maximum out-degree $z$.

We shall call $M_{d,z,k}$ *the mixed Moore bound* for mixed graphs of maximum degree $d$, maximum out-degree $z$ and diameter $k$. A mixed graph of maximum degree $d$, maximum out-degree $z$, diameter $k$ and order $M_{d,z,k}$ is called a *mixed Moore graph*.

Note that $M_{d,z,k} = M_{d,k}$ when $z = 0$ and $M_{d,z,k} = M^*_{d,k}$ when $d = z$.

We summarize the above results in the following theorem.

**Theorem 5.** *Let G be a mixed graph. Then the following two assertions are equivalent.*

(i) *G is a mixed Moore graph of maximum degree $d$, maximum out-degree $z$, diameter $k$ and order $M_{d,z,k}$;*
(ii) *G is a mixed tied graph of maximum degree $d$, maximum out-degree $z$, diameter $k$ and order $M_{d,z,k}$.*

Note that proper mixed Moore graphs do not exist for $k \geq 3$ (see [9]). For $k = 2$, the known proper mixed Moore graphs are the Kautz digraphs and the Bosák graph (see Fig. 2). However, the problem concerning the existence of many proper mixed Moore graphs of $k = 2$ still remains open (see [3,9]). Since mixed Moore graphs of diameter $k \geq 3$ do not exist, it is natural to ask the following question.

**Problem 1.** Construct (totally) regular proper mixed graphs of degree $d \geq 2$, out-degree $z \geq 1$, diameter $k \geq 3$, with the largest possible number of vertices.

## References


[1] E. Bannai, T. Ito, On finite Moore graphs, J. Fac. Sci. Tokyo Univ. 20 (1973) 191–208.
[2] J. Bosák, On the $k$-index of graphs, Discrete Math. 1 (2) (1971) 133–146.
[3] J. Bosák, Partially directed Moore graphs, Math. Slovaca 29 (1979) 181–196.
[4] J. Bosák, A. Kotzig, S. Znám, Strongly geodetic graphs, J. Combin. Theory 5 (1968) 170–176.

[5] W.G. Bridges, S. Toueg, On the impossibility of directed Moore graphs, J. Combin. Theory Ser. B 29 (1980) 339–341.
[6] R.M. Damerell, On Moore graphs, Proc. Cambridge Philos. Soc 74 (1973) 227–236.
[7] A.J. Hoffman, R.R. Singleton, On Moore graphs with diameters 2 and 3, IBM J 4 (1960) 497–504.
[8] M. Miller, J. Širáň, Moore graphs and beyond: A survey of the degree/diameter problem, Electron. J. Combin. DS14 (2005) 1–61.
[9] M.H. Nguyen, M. Miller, J. Gimbert, On mixed Moore graphs, Discrete Math. 307 (2007) 964–970.
[10] J. Plesník, Š. Znám, Strongly geodetic directed graphs, Acta F. R. N. Univ. Comen. - Math. XXIX (1974) 29–34.
[11] D. West, Introduction to Graph Theory, second edition, Prentice Hall, Inc, Upper Saddler River, NJ, 2001.